Instance Optimal Geometric Algorithms

Notes on Tim Roughgarden's *Beyond Worst-Case Analysis*, Lecture 2

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UBC Algorithms Reading Group - May 19th, 2020

Presentation Outline

- Typically we analyze an algorithm where its inputs are parameterized **only** by their sizes.
- By parameterizing the input in more ways, the analysis of the algorithm can be more informative.

- We will discuss the *2D Maxima* problem, which is closely related to the *2D Convex Hull* problem. We analyze the *Kirkpatrick-Seidel (KS)* algorithm in three ways to give the upper bounds:

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  - $O(\min_{S_1,\ldots S_k} \{\sum_{i=1}^{k} |S_i| log \frac{n}{|S_i|}\})$ where $S_1, \ldots S_k$ is a *legal partition* of the input set.

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- We also mention matching lower bounds for each analysis.

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- Suppose algorithms A and B solve the same problem - in what ways can we say A is better than B? In what way can we say A is better than any other algorithm that solves the problem?

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- Problem - too strong: consider *BogoSort* and *BubbleSort* for Z being sorted.

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- *Instance Optimality:* Let $\mathcal{C}$ be a set of algorithms we are interested in comparing algorithm A against. Then we say that A is instance optimal, wrt. approximation-constant $c \geq 1$ and set $\mathcal{C}$, if for all $B \in \mathcal{C}$ and problem instances z,

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- If A is instance optimal, then there is no reason to use any other algorithm for the problem!

Showing instance optimality

- To show A is instance optimal, we need to show two things:
  1. An **upper bound** on A for all instances Z (i.e. $cost(A, Z) \leq x$), and
  2. A **matching lower bound**, up to some constant, for all $B \in \mathcal{C}$ and Z. (i.e. $x \leq c \cdot cost(B, Z)$).

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- **Note:** The matching bound needs to hold for all instances Z. This differs from worst-case analysis, where the bound only needs to match for sufficiently large inputs (i.e. $cost(A) \leq c \cdot cost(B)$ for $n \geq n_0$.)

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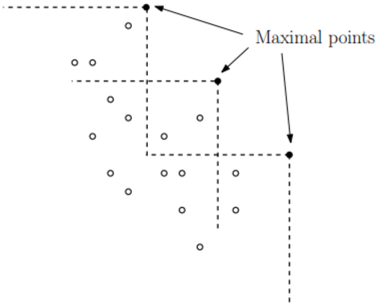
- A *maximal point* is a point not dominate by any others.
- *2DMaxima Problem:* Given point set S, find all maximal points of S.



The Kirkpatrick-Seidel (KS) Algorithm for 2DMaxima

Input: A point set Q
Output: Maximal point set S

1. If $|Q| \leq 1$ add Q to S, return.
2. Compute median x-coordinate among points in Q; partition Q into left and right halves Q_l and Q_r.
3. Let q be the point with max. y-coord. in Q_r. Add q to output set S.
4. Remove q and all points that it dominates (in both Q_l, Q_r.)
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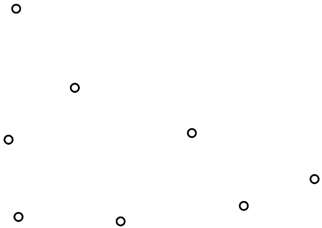

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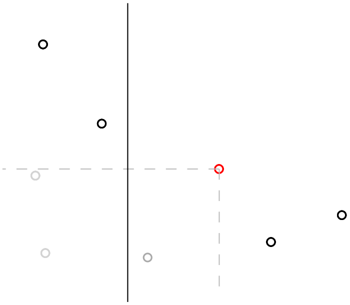

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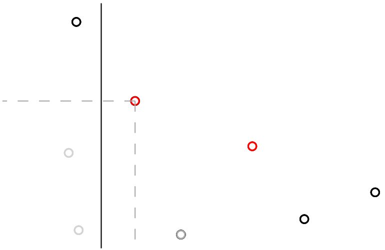

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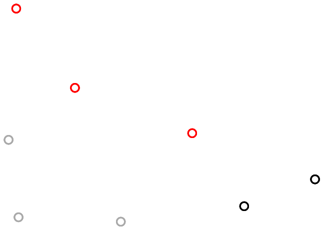

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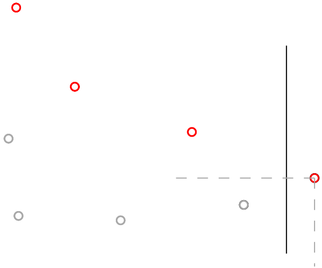

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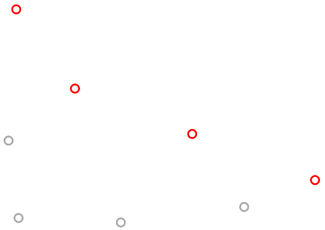

Correctness of KS

- Point q is maximal in the input Q: its x-coord. is larger than all points in Q_l and its y-coord. is larger than all points in Q_r. Clearly, removal of any points dominated by q is correct as well.
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  - p is not dominated by q or any point in Q_l (p has larger x-coord.)
- For maximal points from the recursive call on Q_l, note that after pruning, all points that remain in Q_l must have larger y-coord. than q (i.e. these points cannot be dominated by q.)

Runtime of KS

- Classic Divide-and-Conquer algorithm. For n points $O(n)$ operations are needed to compute median (Blum et al. 1973). Two recursive calls are made. Thus the recurrence is:

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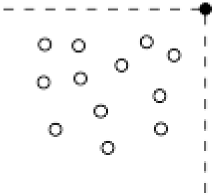
- Thus $T(n) \in \Theta(nlogn)$ - are we not done??

Output-Sensitive Analysis

- Some instances are *easier* than other instances:



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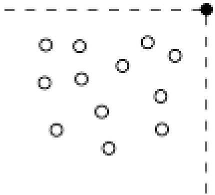

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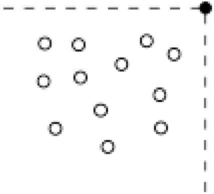
- What makes this instance easy?



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- Input size does not cut it alone! Let's parameterize the input by both number of points n and number of maximal points (i.e. output size) h.
- Claim: The KS algorithm runs in $O(nlogh)$. Proof:
  - Define our recurrence as $T(n, h)$. Let h_l and h_r denote the number of maximal points in the left and right partitions (before removal). Thus we have,

$$T(n,h) \leq \max_{h_l+h_r=h} \{T(\frac{n}{2}, h_l) + T(\frac{n}{2}, h_r)\} + cn$$

    where $h_l, h_r < h$. We proceed by induction.

Proof Continued

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&\leq cnlog(h)
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A more Fine-Grained Analysis

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- For $h \in O(1)$, the runtime of KS is linear. For $h \in O(n)$, the runtime is $nlog(n)$.
- But even for many h in between the algorithm preforms quite-well!
- Why? Many points are dominated by q and removed, resulting in fewer points for recursive calls.
- To explore this more, we need to parameterize the input even further.

Legal Partitions

- To show the instance optimality of the KS algorithm, we use the following parameterization. For partition $S_1, \cdots, S_k$ of input set S, the partition $\{S_i\}$ is a *legal partition/set* if:
  1. S_i contain a single point, or
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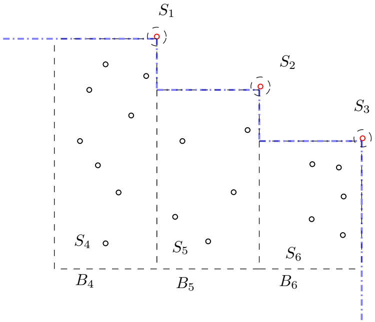

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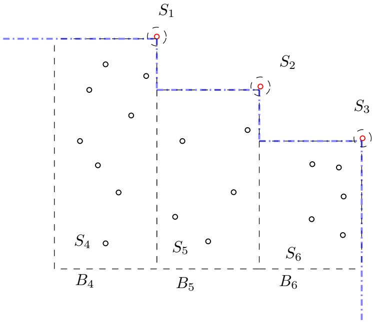
- Intuition: For case 2) if the top-right corner of B_i is a point of the set; choosing this point in KS will remove the entirety of S_i.



Instance Optimal Upper Bound on KS Algorithm

- For a point set S partitioned into k legal sets, the runtime of the KS algorithm is:

$$O\left(\sum_{i=1}^{k} |S_i| log \frac{n}{|S_i|}\right)$$

- What this says: there is a relationship between legal partitions and the rate at which points are removed.

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$$\leq \left(\underbrace{|S_i| + \cdots + |S_i|}_{log(n/|S_i|)+1} + \frac{|S_i|}{1} + \frac{|S_i|}{2^1} + \frac{|S_i|}{2^2} + \cdots \right)$$

Proof Continued

$$\leq |S_i|\Big(log(n/|S_i|) + 3\Big)$$

- which is in $O(|S_i|log(n/|S_i|))$. As each S_i is a partition of the input set, each of the k partitions contributes $O\left(\sum_{i=1}^{k} |S_i| log \frac{n}{|S_i|} \right)$ to the algorithm.

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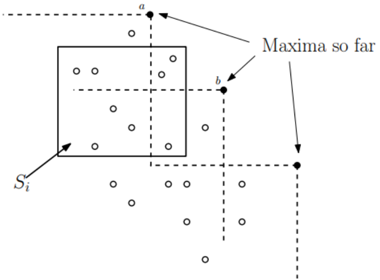

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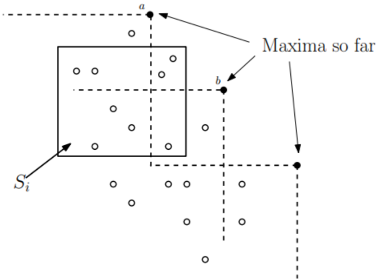

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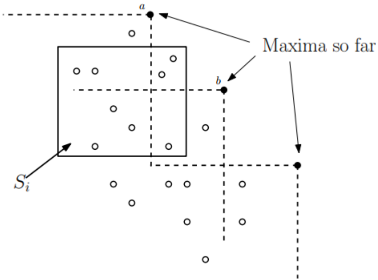
- B_i (and thus S_i) is below the staircase of S - as a is maximal, all points of S_i have y-coord less than a.



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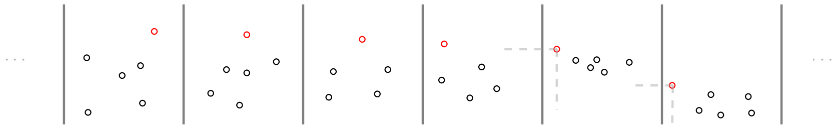
- At level j we've partitioned the point set into at most 2^j (non-empty) *buckets*. In each bucket, there are at most $n/2^j$ points.



Proof of Claim Continued

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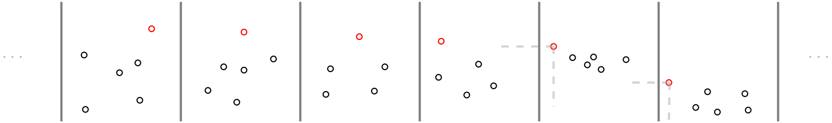
- Each recursive call identifies a maximal point. Once identified and removed, at most $2n/2^j$ points can remain between consecutive buckets.



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- **Note:** When each S_i is a singleton, we have the $O(nlogn)$ bound, and,
- when each maximal point is a singleton, and non-maximal points are in sets below and left of each maximal point, we have the $O(nlogh)$ bound.

A Matching Lower Bound for Instance Optimality?

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- But, *"annoying counterexamples are not a good reason to abandon the quest for an interesting theorem"* - Tim Roughgarden.

What to do?

- There are two approaches:
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- There are two approaches:
  1. Restrict algorithms B to be *order-oblivious*; the input set Q must first be sorted to compare it against hard coded Z.
  2. Redefine $cost(B, Z)$; compare the KS algorithm against the performance of B on permutations of Z - take the max or average of this cost.

A Matching Lower Bound for Instance Optimality

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- **Proof outline:** For any correct algorithm A with input S, there exists a permutation of S on which at least $\Omega\Big(\min_{legal\{S_i\}} \sum_{i=1}^{k} |S_i| log \frac{n}{|S_i|}\Big)$ comparisons are made.

Proof Outline Continued

- A k-d tree ($k = 2$) of axis-aligned boxes is generated - the root is the entire plane, internal nodes are regions (boxes) that are across the staircase of S, and leaf nodes are boxes strictly below the staircase or singletons.

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- An adversary can simulate running A on S and see how permuting the order in which points of S are considered will *hide* maximal points, requiring more comparisons.
- Let D be the sum of the depths of boxes B_p for each $p \in S$ and T be the number of comparisons made by A. It is shown that $T \in \Omega(D)$, and that D is of order $\min\limits_{legal\{S_i\}} \sum_{i=1}^{k} |S_i| log \frac{n}{|S_i|}$.

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  - This was the first result of the paper Afshani, Barbay, and Chan, which can be extended to the 3DMaxima problem and 2D and 3D Convex Hull problem.
- The KS algorithm for 2DMaxima is instance optimal when compared against algorithms that do not "memorize" solution for some inputs.

Conclusions and Take-Away Points

- Instance optimality is *very strong*, and seems problem specific.
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- Instance optimality may not exist for all problems - the best algorithm may rely on the input domain.
- Even if instance optimaility may exist, a matching lower bound needs to be shown on an input-by-input basis. If lower bound proof techniques in the computational model used are not well known, it is difficult to prove such results.

Conclusions and Take-Away Points

- Thank you for listening!
- Next week: *Online Paging and Resource Augmentation*